

Cosmology in rotation-invariant massive gravity with non-trivial fiducial metric

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We investigate the cosmology of $SO(3)$ -invariant massive gravity with 5 degrees of freedom. In contrast with previous studies, we allow for a non-trivial fiducial metric, which can be justified by invoking, for example, a dilaton-like global symmetry. We write the homogeneous and isotropic equations of motion in this more general setup and identify, in particular, de Sitter solutions. We then study the linear perturbations around the homogeneous cosmological solutions, by deriving the quadratic Lagrangian governing the dynamics of scalar, vector and tensor modes. We thus obtain the conditions for the perturbations to be well-behaved. We show that it is possible to find de Sitter solutions whose perturbations are weakly coupled and stable, i.e. without ghost-like or gradient instabilities.

I. INTRODUCTION

Open questions in modern cosmology such as the origin of the accelerating universe or of the flattening of galaxy rotation curves have provided strong motivation to study modifications of gravity in the infrared (IR). While, in the usual explanation, dark energy and dark matter are responsible for those interesting phenomena, it is also natural to ask whether one can change the behavior of gravity in the IR from general relativity (GR) to address these mysteries.

One possibility to modify gravity in the IR is to give a mass to the graviton. Theoretically speaking, however, it has been a long standing problem whether the graviton can have a non-vanishing mass. While Fierz and Pauli (FP) in 1939 [1] found the unique Lorentz-invariant linear theory of massive gravity without ghost instability in Minkowski background, it was found in 1970 by van Dam, Veltman and Zakharov (vDVZ) [2, 3] that the massless limit of the FP theory does not reproduce the massless theory, i.e. GR. Indeed, the post-Newtonian parameter γ in the massless limit is $1/2$ and does not agree with its GR value of 1 . Hence, if the linear theory is valid in the massless limit then massive gravity would have been experimentally excluded, however small the graviton mass is. In 1972 Vainshtein [4] then showed that the linear approximation breaks down in the massless limit and thus the vDVZ discontinuity is not physical. He further argued that because of nonlinearity the massless limit of massive gravity recovers GR in the region where nonlinearity is significant enough. The radius of this region for a given gravitational source is often called the Vainshtein radius, and can be made arbitrarily large by making the graviton mass sufficiently small. However, in the same year Boulware and Deser [5] showed that nonlinear extensions of the FP theory generically exhibit ghost instability. (See reviews [6, 7] for more details of the history and various issues of massive gravity.) Since then, the theory of massive gravity has been haunted by this ghost, often called the Boulware-Deser (BD) ghost, for almost 40 years until de Rham, Gabadadze and Tolley (dRGT) [8, 9] finally found, in 2010, a fully nonlinear theory of massive gravity without BD ghost. While the dRGT theory was constructed in such a way that it avoids the BD ghost in the so-called decoupling limit, the general proof of the absence of BD ghost away from the decoupling limit was completed by Hassan and Rosen [10, 11] (see also [12]).

Having a promising candidate for a theoretically consistent nonlinear theory of massive gravity, it is natural to investigate its cosmological implications. However, it was soon uncovered that the original dRGT theory does not admit any nontrivial flat homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) solution [13]. While this no-go result extends to the closed FLRW case, self-accelerating open FLRW solutions were found in [14]. The negative spatial curvature of this solution can be made arbitrarily close to zero, and current observational data are consistent with a negative curvature of a percent level. Moreover, by replacing the Minkowski fiducial metric¹ in the dRGT theory with a de Sitter or FLRW fiducial metric, one can find both self-accelerating [15] and non self-accelerating [16–18] solutions of not only open but also flat and closed FLRW types. Unfortunately, it was later found in [19] that all homogeneous and isotropic FLRW solutions in dRGT theory are unstable due to either a linear ghost called Higuchi ghost [20] or a new type of nonlinear ghost. The new nonlinear ghost instability stems from the fact

¹ The covariant formulation of the dRGT theory has a physical metric and four Stückelberg scalar fields. The pullback of the metric in the space of Stückelberg scalar fields into the physical spacetime is called the fiducial metric.

that three among five physical degrees of freedom have vanishing (time) kinetic terms on the self-accelerating FLRW solutions [15].

In principle this no-go result can be circumvented by breaking either homogeneity or isotropy. In [13] inhomogeneous cosmological solutions (see also [21–26] for related solutions) were considered and it was shown that the standard cosmological evolution in GR can be recovered by making the Vainshtein radius as large as the size of the cosmological horizon. On the other hand, it was found in [27] that the dRGT theory with the de Sitter or FLRW fiducial metric admits attractor solutions with broken isotropy. On the attractor called anisotropic FLRW solution, at the level of background, the anisotropy is entirely included in the relation between physical and fiducial metrics and is thus hidden from observers probing the physical metric. The stability of anisotropic FLRW solutions was investigated in [28].

An alternative possibility to evade the no-go result is to extend the theory by introducing an extra degree(s) of freedom. For example, one can promote each coupling constant in the dRGT action to a function of an extra scalar field [29, 30]. One can also multiply the fiducial metric of the dRGT theory with a conformal factor that is a function of an extra scalar field. The quasi-dilaton theory [29] is an example of this type and admits a self-accelerating FLRW ² de Sitter solution. While the self-accelerating de Sitter solution in the original quasi-dilaton theory always exhibits ghost instability [31, 32] ³, it is stable in a regime of parameters in an extended version of the quasi-dilaton, where not only conformal but also disformal transformations are applied to the fiducial metric [34, 35].

A disformal transformation of the fiducial metric was first considered in the context of massive gravity coupled to a DBI Galileon (DBI massive gravity) [36]. However, on all known homogeneous and isotropic self-accelerating solutions in the DBI massive gravity, three among the six degrees of freedom have vanishing (time) kinetic terms [37, 38] and are thus expected to exhibit nonlinear instability as in the case of dRGT.

In this work, we explore yet another way to evade the no-go result and to have stable cosmological solutions in a massive gravity theory. In cosmological applications of massive gravity theories the graviton mass is often set to be of the order of the Hubble expansion rate H of the universe. Considering the fact that a non-vanishing H spontaneously breaks Lorentz invariance and respects $SO(3)$ -invariance only, it is natural to require that the graviton mass of order H should be related to $SO(3)$ -invariance but not necessarily the full Lorentz invariance. This consideration leads to yet another possibility to extend the dRGT theory. Namely, we divide the 4-dimensional fiducial metric into a 3-dimensional spatial fiducial metric and a 1-form corresponding to the derivative of the temporal Stückelberg field, and then treat them as independent ingredients in the action. This type of massive gravity with $SO(3)$ -invariance is not new and has been considered in the literature [39–41]. A general $SO(3)$ -invariant massive gravity with five degrees of freedom was recently constructed at fully nonlinear level in [42, 43]. Cosmology in this general framework, with some restrictions, was then investigated in [44] and it was shown that the (time) kinetic terms of the five degrees of freedom generically do not vanish on FLRW backgrounds but that three among the five again have vanishing (time) kinetic terms on de Sitter backgrounds.

The purpose of the present paper is to revisit cosmology in the $SO(3)$ -invariant massive gravity of [42, 43] in a setup that is more general than in [44]. Contrary to the case in [44], we find that all five degrees of freedom can have non-vanishing kinetic terms on strictly homogeneous and isotropic de Sitter backgrounds. In particular, we provide a simple class of concrete models that admit de Sitter solutions on which all 5 degrees of freedom are weakly coupled and stable.

The rest of the present paper is organized as follows. In the next section, we motivate and present the model, first in a covariant formulation, then in the unitary gauge. In section III, we consider the homogeneous and isotropic solutions and derive the Friedmann equations. In the subsequent section, section IV, we turn to the study of the linear perturbations around the homogeneous and isotropic solutions and derive the conditions for stability. In section V, we introduce a simple ansatz for the potential of massive gravity characterized by a few parameters and apply to it the results of the previous sections. Section VI is devoted to a brief comparison of our results with previous studies and we conclude in the final section.

II. THE MODEL

In the present paper we study cosmology in $SO(3)$ -invariant massive gravity. We first motivate and describe the setup before going into the details of the analysis in the following sections.

² Here, FLRW indicates that both physical and fiducial metrics are homogeneous and isotropic.

³ In the decoupling limit of the original quasi-dilaton, another type of self-accelerating solution was recently found and claimed to be stable [33].

A. Motivation

The attempts to modify gravity in the IR can be divided, roughly speaking, into two categories. One is to change the theory itself from GR. A typical example in this category is massive gravity, giving a non-vanishing mass to the spin-2 graviton. The other is to change the state of the universe without changing GR itself. This is the idea of Higgs phase of gravity, and the simplest illustration is the ghost condensate [45], in which a Nambu-Goldstone boson associated with spontaneously broken time diffeomorphism invariance is “eaten” by the graviton and modifies the IR behavior of gravity. An advantage of the latter approach is that the structure of the low-energy effective field theory is robustly determined by the symmetry breaking pattern.

While massive gravity was recently promoted to a fully nonlinear theory at least classically [8, 9],⁴ the regime of validity of the effective field theory is very limited because of a rather low cutoff scale $\Lambda_3 \simeq (m_g^2 M_{\text{Pl}})^{1/3}$, where m_g is the graviton mass. It is thus favorable to seek a possible (partial) UV completion. Although a (partial) UV completion has not yet been found, one ideal possibility may be to realize massive gravity at low energy as a consequence of spontaneous symmetry breaking, i.e. due to an analogue of the Higgs mechanism in gravity. This would unify the two approaches to IR modification of gravity described above.

With this in mind, it seems natural to expect that the structure of the low-energy effective field theory of massive gravity may depend on the choice of background since different backgrounds generically have different symmetry breaking patterns. A background of particular interest is the cosmological one. Since a generic FLRW background respects the 3-dimensional maximal symmetry but breaks the 4-dimensional maximal symmetry, in the present paper we shall consider a class of massive gravity theories which have the same symmetry type. Furthermore we shall restrict our considerations to those with five physical degrees of freedom, the same number of degrees of freedom as a Lorentz-invariant massive spin-2 field in Minkowski background. Fortunately, a general class of theories respecting the 3-dimensional maximal symmetry and with five degrees of freedom was recently found in [42, 43]. In the present paper we thus investigate cosmology in this class of theories.

B. Covariant description

General massive gravity with 3-dimensional Euclidean symmetry has a covariant description in terms of a 4-dimensional physical metric $g_{\mu\nu}$ and four scalar fields (Φ, Φ^I) ($I = 1, 2, 3$). In this covariant description, the action enjoys the 4-dimensional diffeomorphism invariance

$$x^\mu \rightarrow x'^\mu(x^\nu), \quad (1)$$

as well as the internal 3-dimensional Euclidean symmetry:

$$\Phi^I \rightarrow \Phi^I + C^I, \quad \Phi^I \rightarrow V_J^I \Phi^J, \quad (2)$$

where C^I ($I = 1, 2, 3$) are arbitrary constants and V_J^I is an arbitrary $SO(3)$ rotation. This global symmetry implies that the fields Φ^I enter the action only through the fiducial metric $e_{\mu\nu}$ defined by

$$e_{\mu\nu} \equiv \delta_{IJ} \partial_\mu \Phi^I \partial_\nu \Phi^J. \quad (3)$$

Note that $e_{\mu\nu}$ is a degenerate symmetric tensor from the 4-dimensional point of view.

On the other hand, there is a priori no restriction on how Φ enters the action. Consequently, the action can be constructed from the following ingredients:

$$g_{\mu\nu}, \quad \nabla_\mu, \quad R^\mu_{\nu\rho\sigma}, \quad \Phi, \quad e_{\mu\nu}, \quad (4)$$

where ∇_μ and $R^\mu_{\nu\rho\sigma}$ are, respectively, the covariant derivative and the Riemann curvature tensor associated with the metric $g_{\mu\nu}$. Since we are interested in a low-energy effective field theory of gravity relevant to late-time cosmology, we shall not consider higher derivative terms and restrict our attention to an action for gravity of the following form (in the Einstein frame):

$$S_{\text{grav}} = M_{\text{Pl}}^2 \int d^4x \sqrt{-g} \left[\frac{R}{2} - m^2 V(g_{\mu\nu}, \Phi, \partial_\mu \Phi, e_{\mu\nu}) \right], \quad (5)$$

⁴ However, see [46] for some evidences of technical naturalness.

where $m (> 0)$ corresponds to the graviton mass, and $V(g_{\mu\nu}, \Phi, \partial_\mu \Phi, e_{\mu\nu})$ is a scalar made of $g_{\mu\nu}$ (and its inverse $g^{\mu\nu}$), Φ , $\partial_\mu \Phi$ and $e_{\mu\nu}$. Generically, a theory of this form contains 6 degrees of freedom, one of which being a BD ghost. In order to obtain a ghost-free theory, we follow the work [42, 43], which constructed in a systematic way potentials leading to ghost-free theories, with only five physical degrees of freedom. We now summarize their results.

In order to write down the action of the theory with 5 (instead of 6) degrees of freedom, it is convenient to introduce a set of 10 scalars $(\mathcal{N}, \mathcal{N}^I, \Gamma^{IJ} = \Gamma^{JI})$ as

$$\mathcal{N} \equiv \frac{1}{\sqrt{-g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi}}, \quad \mathcal{N}^I \equiv \mathcal{N} n^\mu \partial_\mu \Phi^I, \quad \Gamma^{IJ} \equiv (g^{\mu\nu} + n^\mu n^\nu) \partial_\mu \Phi^I \partial_\nu \Phi^J, \quad (6)$$

where we have defined the unit vector

$$n^\mu \equiv \mathcal{N} g^{\mu\nu} \partial_\nu \Phi. \quad (7)$$

We also introduce a set of 3 auxiliary scalars ξ^I ($I = 1, 2, 3$) and then define 6 additional scalars \mathcal{K}^{IJ} ($= \mathcal{K}^{JI}$) ($I, J = 1, 2, 3$) by

$$\mathcal{K}^{IJ} \equiv \Gamma^{IJ} - \xi^I \xi^J. \quad (8)$$

The theory is then characterized by two arbitrary $SO(3)$ -invariant functions

$$\mathcal{U}(\mathcal{K}^{IJ}, \delta_{IJ}, \Phi) \quad \text{and} \quad \mathcal{E}(\Gamma^{IJ}, \xi^I, \delta_{IJ}, \Phi). \quad (9)$$

To write down the potential V explicitly, we first impose the following three constraints relating ξ^I to \mathcal{N}^I ($I = 1, 2, 3$):

$$\mathcal{N}^I - \mathcal{N} \xi^I = -\mathcal{U}^{IJ} \mathcal{E}_J, \quad (10)$$

where \mathcal{U}^{IJ} is the inverse of the Hessian matrix $\mathcal{U}_{IJ} \equiv \partial^2 \mathcal{U} / \partial \xi^I \partial \xi^J$ and $\mathcal{E}_J \equiv \partial \mathcal{E} / \partial \xi^J$. The potential V is then specified as

$$V = \mathcal{U} + \frac{\mathcal{E} - \mathcal{U}_I \mathcal{U}^{IJ} \mathcal{E}_J}{\mathcal{N}}, \quad (11)$$

where $\mathcal{U}_I \equiv \partial \mathcal{U} / \partial \xi^I$. It has been shown in the unitary gauge that the form of the potential term V , specified by (11), guarantees to suppress one of the original 6 degrees of freedom, leaving only 5 physical degrees of freedom in the theory [42, 43]. This will also be manifest in our linear analysis (and discussed in particular in Appendix B).

C. Fiducial metric

The general theory is characterized by two $SO(3)$ -invariant functions $\mathcal{U}(\mathcal{K}^{IJ}, \delta_{IJ}, \Phi)$ and $\mathcal{E}(\Gamma^{IJ}, \xi^I, \delta_{IJ}, \Phi)$. While the $SO(3)$ -invariance restricts the way these two functions depend on Γ^{IJ} and ξ^I , the dependence on Φ is arbitrary. This arbitrariness significantly reduces the predictability of the theory. In the following we thus consider two specific cases with additional global symmetries that restrict the way Φ can enter $\mathcal{U}(\mathcal{K}^{IJ}, \delta_{IJ}, \Phi)$ and $\mathcal{E}(\Gamma^{IJ}, \xi^I, \delta_{IJ}, \Phi)$. Motivated by this, we then propose a prescription that is general enough to cover the two cases specified by symmetries and that is still simple enough for the analysis in the forthcoming subsections.

1. Case with shift symmetry

One of the simplest symmetries one can envisage is the shift symmetry, i.e.

$$\Phi \rightarrow \Phi + C, \quad (12)$$

where C is an arbitrary constant. In this case, the two functions are of the form

$$\mathcal{U} = \mathcal{U}(\mathcal{K}^{IJ}, \delta_{IJ}), \quad \mathcal{E} = \mathcal{E}(\Gamma^{IJ}, \xi^I, \delta_{IJ}). \quad (13)$$

Theories of this type however do not admit FLRW solutions, unless the function \mathcal{E} is fine-tuned [44]. (As a special case, this class of theories includes the original dRGT theory with Minkowski fiducial metric, which does not allow for the FLRW cosmology.) Therefore, we will not explore further this class of theories.

2. Case with dilaton-like symmetry

As a simple deformation of the shift symmetry, we now consider the following global symmetry:

$$\Phi \rightarrow \Phi + C, \quad \Phi^I \rightarrow e^{-MC} \Phi^I, \quad (14)$$

where C is an arbitrary constant and M some energy scale. In this case, the two functions \mathcal{U} and \mathcal{E} are of the form

$$\mathcal{U} = \tilde{\mathcal{U}}(\tilde{\mathcal{K}}^{IJ}, \delta_{IJ}), \quad \mathcal{E} = \tilde{\mathcal{E}}(\tilde{\Gamma}^{IJ}, \tilde{\xi}^I, \delta_{IJ}), \quad (15)$$

where we have introduced the rescaled quantities

$$\tilde{\Gamma}^{IJ} \equiv e^{2M\Phi} \Gamma^{IJ}, \quad \tilde{\xi}^I \equiv e^{M\Phi} \xi^I, \quad \tilde{\mathcal{K}}^{IJ} \equiv e^{2M\Phi} \mathcal{K}^{IJ} = \tilde{\Gamma}^{IJ} - \tilde{\xi}^I \tilde{\xi}^J, \quad (16)$$

which are invariant under the transformation (14).

The constraints (10) and the potential (11) are expressed in terms of them as

$$\tilde{\mathcal{N}}^I - \mathcal{N} \tilde{\xi}^I = -\tilde{\mathcal{U}}^{IJ} \tilde{\mathcal{E}}_J, \quad (17)$$

and

$$V = \tilde{\mathcal{U}} + \frac{\tilde{\mathcal{E}} - \tilde{\mathcal{U}}_I \tilde{\mathcal{U}}^{IJ} \tilde{\mathcal{E}}_J}{\mathcal{N}}, \quad (18)$$

where $\tilde{\mathcal{U}}^{IJ}$ is the inverse of the Hessian matrix $\tilde{\mathcal{U}}_{IJ} \equiv \partial^2 \tilde{\mathcal{U}} / \partial \tilde{\xi}^I \partial \tilde{\xi}^J$, $\tilde{\mathcal{U}}_I \equiv \partial \tilde{\mathcal{U}} / \partial \tilde{\xi}^I$, $\tilde{\mathcal{E}}_J \equiv \partial \tilde{\mathcal{E}} / \partial \tilde{\xi}^J$, and $\tilde{\mathcal{N}}^I \equiv e^{M\Phi} \mathcal{N}^I$.

3. Prescription with general fiducial metric

Motivated by the two specific cases considered above, we now propose a class of theories specified by the two $SO(3)$ -invariant functions \mathcal{U} and \mathcal{E} of the form

$$\mathcal{U} = \mathcal{U}(\mathcal{K}^{IJ}, f_{IJ}), \quad \mathcal{E} = \mathcal{E}(\Gamma^{IJ}, \xi^I, f_{IJ}), \quad (19)$$

where

$$f_{IJ} = b^2(\Phi) \delta_{IJ}, \quad (20)$$

is the fiducial three-dimensional metric including the dependence on the scalar field Φ . The auxiliary scalars ξ^I and the potential V are then constructed as in (10) and (11).

This prescription includes the two particular cases discussed above: $b(\Phi) = 1$ for the case with shift symmetry; and $b(\Phi) = e^{M\Phi}$ for the case with dilaton-like symmetry. In the forthcoming sections, we treat $b(\Phi)$ as a general positive function of Φ , which can be seen as a scale factor in the 3-d field space spanned by the Φ^I , parametrized by Φ .

D. ADM formulation in the unitary gauge

For simplicity, it is convenient to work directly in the unitary gauge, i.e. in the privileged coordinate system associated with the preferred slicing of the theory, such that

$$\Phi = t, \quad \Phi^i = x^i, \quad (21)$$

and to write the metric in the ADM form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (22)$$

where N denotes the lapse function and N^i the shift vector. In this framework, the theory with general fiducial metric is specified by the two scalars

$$\mathcal{U} = \mathcal{U}(\mathcal{K}^{ij}, f_{ij}), \quad \mathcal{E} = \mathcal{E}(\gamma^{ij}, \xi^i, f_{ij}), \quad (23)$$

where γ^{ij} is the inverse of the spatial metric γ_{ij} ,

$$f_{ij} = b^2(t)\delta_{ij}, \quad (24)$$

is the fiducial three-dimensional metric in the unitary gauge, and the three-dimensional vector ξ^i and the three-dimensional symmetric tensor \mathcal{K}^{ij} are defined through

$$N^i - N\xi^i = -\mathcal{U}^{ij}\mathcal{E}_j, \quad \mathcal{K}^{ij} = \gamma^{ij} - \xi^i\xi^j. \quad (25)$$

Here, \mathcal{U}^{ij} is the inverse of the Hessian matrix $\mathcal{U}_{ij} \equiv \partial^2\mathcal{U}/\partial\xi^i\partial\xi^j$, and $\mathcal{E}_j \equiv \partial\mathcal{E}/\partial\xi^j$. The potential V in the unitary gauge is then specified as

$$V(N, N^i, \gamma^{ij}, f_{ij}) = \mathcal{U} + \frac{\mathcal{E} - \mathcal{U}_i\mathcal{U}^{ij}\mathcal{E}_j}{N}, \quad (26)$$

where $\mathcal{U}_i \equiv \partial\mathcal{U}/\partial\xi^i$. The action in the gravity sector is then of the form

$$S_{\text{grav}} = S_{\text{EH}} + S_{\text{mg}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x N \sqrt{\gamma} \left(K_{ij}K^{ij} - K^2 + {}^{(3)}R \right) - M_{\text{Pl}}^2 m^2 \int d^4x N \sqrt{\gamma} V(N, N^i, \gamma^{ij}, f_{ij}), \quad (27)$$

where $\gamma \equiv \det(\gamma_{ij})$, and “EH” and “mg” denote the Einstein-Hilbert and massive gravity terms, respectively. The first part of the action corresponds to the ADM expression of the Einstein-Hilbert action, where

$$K_{ij} \equiv \frac{1}{2N} (\dot{\gamma}_{ij} - N_{i|j} - N_{j|i}), \quad K \equiv \gamma^{ij}K_{ij} \quad (28)$$

are the extrinsic curvature tensor and its trace (the symbol $|$ denotes the spatial covariant derivative associated with the spatial metric γ_{ij}), while ${}^{(3)}R$ denotes the three-dimensional Ricci scalar associated with the metric γ_{ij} . The second part of the action consists of the massive gravity potential in the unitary gauge, which in general depends explicitly on the time t through the fiducial metric f_{ij} .

III. HOMOGENEOUS COSMOLOGY

Let us now consider the cosmology of this class of theories. It is natural to assume that the preferred frame associated with the massive gravity potential coincides with the cosmological frame (i.e. comoving observers are at rest in the preferred frame). Assuming a spatially flat FLRW spacetime, the physical metric is thus given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\bar{N}^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (29)$$

i.e.

$$\bar{N} = \bar{N}(t), \quad \bar{N}^i = \bar{\xi}^i = 0, \quad \bar{\gamma}_{ij} = a^2(t)\delta_{ij}. \quad (30)$$

In the background, the potential reduces to

$$\bar{V} = \bar{\mathcal{U}} + \frac{\bar{\mathcal{E}}}{\bar{N}}, \quad (31)$$

where an overbar denotes that the corresponding quantity is evaluated on the background. Adding the matter action S_m , the total action on the background is thus given by

$$\bar{S}_{\text{total}} = \bar{S}_{\text{grav}} + \bar{S}_m = M_{\text{Pl}}^2 \int d^4x a^3 \left(-3 \frac{\dot{a}^2}{\bar{N}a^2} - m^2 \bar{N} \bar{\mathcal{U}} - m^2 \bar{\mathcal{E}} \right) + \bar{S}_m, \quad (32)$$

where an overdot represents a derivative with respect to the time t .

Both $\bar{\mathcal{U}}$ and $\bar{\mathcal{E}}$ depend on the scale factors a and b , via their dependence on γ^{ij} and f_{ij} respectively. More precisely, they depend on the ratio

$$X \equiv \frac{b}{a}, \quad (33)$$

since $\bar{\mathcal{U}}$, like $\bar{\mathcal{E}}$, depends on scalar combinations of the matrix

$$\bar{\gamma}^{ik} f_{kj} = \frac{b^2}{a^2} \delta_j^i = X^2 \delta_j^i. \quad (34)$$

where $\bar{\gamma}^{ij} = \delta^{ij}/a^2(t)$ is the inverse of $\bar{\gamma}_{ij} = a^2(t)\delta_{ij}$. Introducing the notations

$$\overline{\left(\frac{\partial \mathcal{U}}{\partial \mathcal{K}^{ij}}\right)} \equiv \mathcal{U}' \bar{\gamma}_{ij}, \quad \overline{\left(\frac{\partial \mathcal{E}}{\partial \gamma^{ij}}\right)} \equiv \mathcal{E}' \bar{\gamma}_{ij}, \quad (35)$$

one immediately finds

$$X \frac{\partial \bar{\mathcal{U}}}{\partial X} = 6\mathcal{U}', \quad X \frac{\partial \bar{\mathcal{E}}}{\partial X} = 6\mathcal{E}'. \quad (36)$$

This implies in particular $a\partial\bar{\mathcal{U}}/\partial a = -6\mathcal{U}'$ and $a\partial\bar{\mathcal{E}}/\partial a = -6\mathcal{E}'$.

The variation of the total homogeneous action \bar{S}_{total} , with respect to \bar{N} and to a , yields the Friedmann equations, which read

$$3M_{\text{Pl}}^2 H^2 = \rho_g + \rho_m, \quad M_{\text{Pl}}^2 \left(2\frac{\dot{H}}{\bar{N}} + 3H^2 \right) = -P_g - P_m, \quad (37)$$

where $H \equiv \dot{a}/(\bar{N}a)$ is the Hubble expansion rate. On the right hand sides, we find the matter energy density and pressure, respectively denoted by ρ_m and P_m , as well as the effective energy density and pressure associated with the massive gravity potential, expressed as

$$\rho_g \equiv M_{\text{Pl}}^2 m^2 \bar{\mathcal{U}}, \quad P_g \equiv M_{\text{Pl}}^2 m^2 \left[2\mathcal{U}' - \bar{\mathcal{U}} + \frac{1}{\bar{N}} (2\mathcal{E}' - \bar{\mathcal{E}}) \right]. \quad (38)$$

Even in the absence of matter, one can obtain an accelerating solution $\partial_t(\dot{a}/\bar{N}) > 0$, provided

$$\rho_g + 3P_g = M_{\text{Pl}}^2 m^2 \left[6\mathcal{U}' - 2\bar{\mathcal{U}} + \frac{1}{\bar{N}} (6\mathcal{E}' - 3\bar{\mathcal{E}}) \right] < 0. \quad (39)$$

The Bianchi identity and the conservation of ordinary matter imply the following conservation equation for the effective gravitational component:

$$\dot{\rho}_g + 3\bar{N}H(\rho_g + P_g) = M_{\text{Pl}}^2 m^2 \left[\dot{\bar{\mathcal{U}}} + 3H(2\bar{N}\mathcal{U}' + 2\mathcal{E}' - \bar{\mathcal{E}}) \right] = 0. \quad (40)$$

The quantity $\bar{\mathcal{U}}$ is in general time-dependent, via its dependence on X , and using (36), one finds

$$\dot{\bar{\mathcal{U}}} = -6 \left(\bar{N}H - \frac{\dot{b}}{b} \right) \mathcal{U}'. \quad (41)$$

Substituting this into (40), one finally obtains the relation,⁵

$$\mathcal{U}' \frac{\dot{b}}{b} + H \left(\mathcal{E}' - \frac{1}{2} \bar{\mathcal{E}} \right) = 0. \quad (42)$$

This constraint equation complements the Friedmann equations given in (37).

In the absence of matter ($\rho_m = P_m = 0$), and assuming expansion ($H > 0$), we have from (37) and (42),

$$H = \frac{m}{\sqrt{3}} \sqrt{\bar{\mathcal{U}}}, \quad \mathcal{U}' \frac{\dot{b}}{b} + \frac{m}{\sqrt{3}} \sqrt{\bar{\mathcal{U}}} \left(\mathcal{E}' - \frac{1}{2} \bar{\mathcal{E}} \right) = 0. \quad (43)$$

⁵ This relation can also be obtained directly from the action by keeping the time-like Stückelberg field Φ and taking the variation of the action with respect to Φ . In this case, \dot{b} is replaced by $\partial_\Phi b$.

Since the ratio \dot{b}/b is not dynamical but fixed by the theory, the second equation above should be considered as an algebraic equation for X . Once this is solved with respect to X , the first equation determines the background geometry.

For example, in the theory with dilaton-like symmetry considered in subsection II C, we have $\dot{b}/b = M$. Hence we obtain

$$\mathcal{U}'M + \frac{m}{\sqrt{3}}\sqrt{\bar{\mathcal{U}}}\left(\mathcal{E}' - \frac{1}{2}\bar{\mathcal{E}}\right) = 0, \quad (44)$$

which is a time-independent algebraic equation for X . Provided that this equation allows for a real solution $X = X_0$, the Hubble expansion rate is determined by

$$H = \frac{m}{\sqrt{3}}\sqrt{\bar{\mathcal{U}}|_{X=X_0}}. \quad (45)$$

Hence the graviton mass term acts as an effective cosmological constant, implying that all vacuum solutions are de Sitter once we impose the dilaton-like symmetry (14). In this example the value of the lapse \bar{N} is set by the model parameters through $\bar{N} = M/H$, assuming $\mathcal{U}' \neq 0$, so that the coordinate time t does not correspond to the usual intuitive cosmic time, i.e. the proper time $\tau = \int \bar{N} dt$. As seen from (44) and (45), changing M implies changing the value of X_0 and thus that of H in a de Sitter vacuum. In this sense, the parameter M , i.e. the dependence of b on Φ , directly influences the relation between H and X .

IV. LINEAR PERTURBATIONS

We now turn to the study of linear perturbations about the homogeneous solutions introduced in the previous section. For simplicity we consider the pure massive gravity system without matter.

Perturbations of the background metric (29), in the ADM form (22), are described by the lapse perturbation δN , defined by

$$N = \bar{N}(t) + \delta N, \quad (46)$$

the shift vector N^i , which is already a perturbation as it vanishes in the background, and the perturbations h_{ij} of the three-dimensional metric, defined by

$$\gamma_{ij} = \bar{\gamma}_{ij} + \delta\gamma_{ij} = a^2(t)(\delta_{ij} + h_{ij}). \quad (47)$$

We expand the total gravitational action up to quadratic order in the perturbations. As shown in Appendix A, the quadratic part of the Einstein-Hilbert action can be written as

$$S_{\text{EH}}^{(2)} = \tilde{S}_{\text{EH}}^{(2)} + M_{\text{Pl}}^2 \int d^4x (N\sqrt{\gamma})^{(2)} 3H^2, \quad (48)$$

where $(N\sqrt{\gamma})^{(2)}$ is the quadratic part of $N\sqrt{\gamma}$ and the explicit expression for $\tilde{S}_{\text{EH}}^{(2)}$ is given in (A8).

On the other hand, the graviton mass term, expanded up to quadratic order, yields

$$\begin{aligned} S_{\text{mg}} = & -M_{\text{Pl}}^2 m^2 \int d^4x \left(N\sqrt{\gamma}\bar{\mathcal{U}} + \sqrt{\gamma}\bar{\mathcal{E}} \right) \\ & -M_{\text{Pl}}^2 m^2 \int d^4x \sqrt{\gamma} \left\{ (N\mathcal{U}' + \mathcal{E}') \bar{\gamma}_{ij} \delta\gamma^{ij} - \left(\bar{N}\mathcal{U}' + \frac{1}{2}\mathcal{E}'' \right) \bar{\gamma}_{ij} \xi^i \xi^j \right. \\ & \left. + \left[\frac{1}{2} (\bar{N}\mathcal{U}_s'' + \mathcal{E}_s'') \bar{\gamma}_{ij} \bar{\gamma}_{kl} + \frac{1}{4} (\bar{N}\mathcal{U}_t'' + \mathcal{E}_t'') (\bar{\gamma}_{ik} \bar{\gamma}_{jl} + \bar{\gamma}_{il} \bar{\gamma}_{jk}) \right] \delta\gamma^{ij} \delta\gamma^{kl} \right\} + O(\epsilon^3), \end{aligned} \quad (49)$$

where ϵ counts the order of the perturbative expansion, and the “second” derivatives of \mathcal{E} and \mathcal{U} are defined as follows:

$$\overline{\left(\frac{\partial^2 \mathcal{U}}{\partial \mathcal{K}^{ij} \partial \mathcal{K}^{kl}} \right)} \equiv \mathcal{U}_s'' \bar{\gamma}_{ij} \bar{\gamma}_{kl} + \frac{1}{2} \mathcal{U}_t'' (\bar{\gamma}_{ik} \bar{\gamma}_{jl} + \bar{\gamma}_{il} \bar{\gamma}_{jk}), \quad (50)$$

$$\overline{\left(\frac{\partial^2 \mathcal{E}}{\partial \gamma^{ij} \partial \gamma^{kl}} \right)} \equiv \mathcal{E}_s'' \bar{\gamma}_{ij} \bar{\gamma}_{kl} + \frac{1}{2} \mathcal{E}_t'' (\bar{\gamma}_{ik} \bar{\gamma}_{jl} + \bar{\gamma}_{il} \bar{\gamma}_{jk}), \quad \overline{\left(\frac{\partial^2 \mathcal{E}}{\partial \xi^i \partial \xi^j} \right)} \equiv \mathcal{E}'' \bar{\gamma}_{ij}. \quad (51)$$

Here, we recall that quantities with overbar are those evaluated on the background. Using

$$\xi^i = \frac{2\mathcal{U}'}{2\bar{N}\mathcal{U}' + \mathcal{E}''} N^i + \mathcal{O}(\epsilon^2), \quad (52)$$

which follows from (25), and

$$\gamma^{ij} = \frac{1}{a^2} \left(\delta^{ij} - h^{ij} + h^{ik} h_k^j \right) + \mathcal{O}(\epsilon^3), \quad (53)$$

the graviton mass term, up to quadratic order, is rewritten in the form

$$\begin{aligned} S_{\text{mg}} = & - \int d^4x N \sqrt{\gamma} \rho_g - M_{\text{Pl}}^2 m^2 \int d^4x a^3 \left\{ \bar{\mathcal{E}} + \left(\frac{1}{2} \mathcal{E} - \bar{N} \mathcal{U}' - \mathcal{E}' \right) h \right. \\ & - \mathcal{U}' \delta N h - \frac{2\mathcal{U}'^2}{2\bar{N}\mathcal{U}' + \mathcal{E}''} a^2 \delta_{ij} N^i N^j + \frac{1}{2} \left[\frac{\bar{\mathcal{E}}}{4} - \bar{N} \mathcal{U}' - \mathcal{E}' + \bar{N} \mathcal{U}_s'' + \mathcal{E}_s'' \right] h^2 \\ & \left. + \left[-\frac{\mathcal{E}}{4} + \bar{N} \mathcal{U}' + \mathcal{E}' + \frac{1}{2} (\bar{N} \mathcal{U}_t'' + \mathcal{E}_t'') \right] h_{ij} h^{ij} \right\} + \mathcal{O}(\epsilon^3), \end{aligned} \quad (54)$$

where ρ_g is defined in (38), $h \equiv \delta^{ij} h_{ij}$ and hereafter all spatial indices are raised and lowered by δ^{ij} and δ_{ij} , respectively. This leads to the following expression for the quadratic part of the graviton mass term:

$$S_{\text{mg}}^{(2)} = \tilde{S}_{\text{mg}}^{(2)} - \int d^4x (N \sqrt{\gamma})^{(2)} \rho_g, \quad (55)$$

with

$$\tilde{S}_{\text{mg}}^{(2)} \equiv M_{\text{Pl}}^2 m^2 \int d^4x \bar{N} a^3 \left(c_0 \frac{\delta N^2}{\bar{N}^2} + c_1 \frac{\delta N}{\bar{N}} h + c_2 \bar{\gamma}_{ij} \frac{N^i N^j}{\bar{N}^2} + c_3 h^2 + c_4 h_{ij} h^{ij} \right), \quad (56)$$

where the coefficients in the expansion are given by

$$\begin{aligned} c_0 &= 0, & c_1 &= \mathcal{U}', & c_2 &= \frac{2\bar{N}\mathcal{U}'^2}{2\bar{N}\mathcal{U}' + \mathcal{E}''}, & c_3 &= -\frac{1}{2} \left(\frac{\bar{\mathcal{E}}}{4\bar{N}} - \mathcal{U}' - \frac{\mathcal{E}'}{\bar{N}} + \mathcal{U}_s'' + \frac{\mathcal{E}_s''}{\bar{N}} \right) \\ c_4 &= \frac{\bar{\mathcal{E}}}{4\bar{N}} - \mathcal{U}' - \frac{\mathcal{E}'}{\bar{N}} - \frac{1}{2} \left(\mathcal{U}_t'' + \frac{\mathcal{E}_t''}{\bar{N}} \right). \end{aligned} \quad (57)$$

We have explicitly added a term proportional to δN^2 in (56), even if this term vanishes in the present case, in order to show that a generic potential will generate all the terms listed in the quadratic action above. As discussed in Appendix B, the vanishing of c_0 is directly related to the absence of the sixth degree of freedom, i.e. BD ghost, in the models under study.

We can then rewrite the quadratic action for the pure massive gravity system as

$$S_{\text{grav}}^{(2)} = \tilde{S}_{\text{EH}}^{(2)} + \tilde{S}_{\text{mg}}^{(2)} + \int d^4x (N \sqrt{\gamma})^{(2)} (3M_{\text{Pl}}^2 H^2 - \rho_g). \quad (58)$$

The last term in (58) vanishes once the background equation is imposed, and it thus makes no contribution to the analysis of perturbations. As usual, it is convenient to decompose the perturbations into scalar, vector and tensor modes. We discuss in turn each of these sectors in the following subsections.

A. Tensor modes

Tensor modes are characterized by $\delta N = 0$, $N^i = 0$ and the transverse traceless condition on h_{ij} :

$$\partial^i h_{ij} = 0, \quad h = 0. \quad (59)$$

We then find that the quadratic action for the tensor modes is given by

$$S_T^{(2)} = M_{\text{Pl}}^2 \int d^4x \bar{N} a^3 \left(\frac{1}{8} \frac{\dot{h}_{ij} \dot{h}^{ij}}{\bar{N}^2} - \frac{1}{8a^2} \partial_k h_{ij} \partial^k h^{ij} - \frac{1}{8} m_T^2 h_{ij} h^{ij} \right), \quad (60)$$

with

$$m_T^2 = 4\frac{\dot{H}}{N} - 8m^2 c_4 = -\frac{2}{M_{\text{Pl}}^2}(\rho_g + P_g) - 8m^2 c_4, \quad (61)$$

where we have used the background equations of motion, (37)-(38), in absence of matter (i.e. with $\rho_m = P_m = 0$), to obtain the last equality. As is clear from (60), the tensor modes always have a healthy kinetic term, and the high-energy limit of their propagation speed is unity.

B. Vector modes

For vector perturbations, the lapse has no vector mode, i.e. $\delta N = 0$, the shift is restricted to be transverse, i.e. $\partial_i N^i = 0$, and the metric is decomposed as

$$h_{ij} = \partial_i E_j + \partial_j E_i, \quad \partial_i E^i = 0. \quad (62)$$

The quadratic action for vector modes reads

$$S_V^{(2)} = M_{\text{Pl}}^2 \int d^4x \frac{a^3}{N} \left(\frac{1}{4} \partial_i \dot{E}_j \partial^i \dot{E}^j + \frac{1}{4} \partial_i N_j \partial^i N^j - \frac{1}{2} \partial_i \dot{E}_j \partial^i N^j - \frac{1}{4} \bar{N}^2 m_T^2 \partial_i E_j \partial^i E^j + \mu^2 a^2 \delta_{ij} N^i N^j \right), \quad (63)$$

with

$$\mu^2 \equiv c_2 m^2. \quad (64)$$

The shift components N^i are not dynamical and can be integrated out as follows. The momentum constraint, obtained by varying the action with respect to the shift, yields the relation

$$N_i = \frac{k^2}{k^2 + 4\mu^2 a^2} \dot{E}_i, \quad (65)$$

in Fourier space. Substituting this back into the action, the final action for the two dynamical modes E_i is given by

$$S_V^{(2)} = M_{\text{Pl}}^2 \int dt d^3k \bar{N} a^3 \left(\frac{k^2 \mu^2 a^2}{k^2 + 4\mu^2 a^2} \frac{|\dot{E}_i|^2}{\bar{N}^2} - \frac{k^2}{4} m_T^2 |E_i|^2 \right). \quad (66)$$

Note that the coefficient of the kinetic term vanishes for $\mu = 0$, i.e. for $\mathcal{U}' = 0$ according to (57) and (64), which suggests a strong coupling problem in this limit.⁶ In the following, in order to avoid this problem as well as ghost-like pathologies related to a negative kinetic sign, we will require μ^2 to be strictly positive, which is equivalent to the condition

$$c_2 > 0. \quad (67)$$

The propagation speed for the vector modes can be read from the action (66), by taking the ratio between the coefficient of the gradient term and that of the kinetic term, which yields

$$c_V^2 = \frac{k^2 + 4\mu^2 a^2}{4k^2 \mu^2} m_T^2. \quad (68)$$

⁶ The kinetic term vanishes when $\mathcal{U}' = 0$, not only for the vector sector but also for the scalar sector, as will be seen in the following subsection. In fact, when one considers the limit in which the present theory becomes equivalent to the dRGT theory, the so-called self-accelerating branch corresponds to the limit $\mathcal{U}' = 0$. It has been shown that this branch of the dRGT theory suffers from a nonlinear ghost due to the vanishing (time) kinetic terms of three out of the five degrees of freedom [19] and our result for $\mathcal{U}' = 0$ is thus consistent with this. Moreover, seen from (41), in the case of a trivial fiducial metric $\dot{b} = 0$, a pure de Sitter solution $\dot{\mathcal{U}} = 0$ necessarily requires $\mathcal{U}' = 0$, leading to the vanishing of the kinetic terms. This is also consistent with the results of [44]. A more elaborate discussion on this issue is given in Section VI.

In the high momentum limit, $k^2/a^2 \gg \mu^2$, this reduces to

$$c_V^2 \simeq \frac{m_T^2}{4\mu^2} \quad (k^2/a^2 \gg \mu^2). \quad (69)$$

One thus concludes that, under the no-ghost condition (67), there is no gradient instability in the vector sector if and only if $m_T^2 > 0$, when matter is negligible.

Before ending this subsection, let us comment on the Minkowski limit $\mathcal{H} \equiv H/M \rightarrow 0$ in the theory with dilaton-like symmetry (14), $b(\Phi) = e^{M\Phi}$. By imposing this symmetry, it follows from (41) that $\bar{N} = 1/\mathcal{H}$, since $\dot{\bar{U}} = 0$ and $\dot{b}/b = M$, and from (42) that $\mathcal{U}' \sim \mathcal{O}(\mathcal{H})$, if $\bar{\mathcal{E}} \sim \mathcal{E}' \sim \mathcal{O}(1)$. Further assuming that the other derivatives of \mathcal{U} are of order $\mathcal{O}(\mathcal{H})$ and that those of \mathcal{E} are of order $\mathcal{O}(1)$, we find that all c_i are of order $\mathcal{O}(\mathcal{H})$, and thus $\mu^2 \sim m_T^2 \sim m^2 \times \mathcal{O}(\mathcal{H})$. Therefore in the limit $\mathcal{H} \rightarrow 0$ the vector modes do not suffer from a pathological, i.e. either infinite or vanishing, propagation speed, which would typically be an indicator of strong coupling at nonlinear level.

C. Scalar modes

Scalar perturbations are described by δN , as well as

$$N^i = \delta^{ij} \partial_j B, \quad h_{ij} = 2C\delta_{ij} + 2\partial_i \partial_j E. \quad (70)$$

The quadratic gravitational action thus depends on the four quantities δN , B , C and E . The quantities δN and B are manifestly non-dynamical and variation of the action with respect to them yields two constraints, corresponding respectively to the Hamiltonian constraint and the momentum constraint. From these two constraints, one can extract the expressions of δN and B in terms of the variables E and C , and substitute them back into the action. This gives an action that depends only on C and E .

In the case $c_0 = 0$, it turns out that the variable C is also non-dynamical and can be integrated out. One finally obtains an action in terms of the only degree of freedom left, of the form

$$S_S^{(2)} = M_{\text{Pl}}^2 \int dt d^3k \bar{N} a^3 \left(\kappa \frac{|\dot{E}|^2}{\bar{N}^2} + \mathcal{M} |E|^2 \right). \quad (71)$$

For simplicity, let us now consider a de Sitter background solution, with constant H . According to (43), X must then be constant, as well as \dot{b}/b , which corresponds to a model with dilaton-like symmetry. Since $\bar{N} = (\dot{b}/b)/H$ is also constant, we conclude that all the coefficients defined in (57) are time-independent. In this case, the coefficient of the kinetic term reduces to

$$\kappa = \frac{m^2 k^4}{\mathcal{D}_E} \left\{ c_1 (-c_1 + 2c_2) \tilde{k}^2 + 3c_2 \left[c_1^2 m^2 + (-3c_1 + 12c_3 + 4c_4) H^2 \right] \right\}, \quad (72)$$

where $\tilde{k} \equiv k/a$ and

$$\mathcal{D}_E \equiv c_2 \tilde{k}^4 + 3 \left[2c_1 c_2 m^2 + (-3c_1 + 12c_3 + 4c_4) H^2 \right] \tilde{k}^2 + 9c_2 m^2 \left[c_1^2 m^2 + (-3c_1 + 12c_3 + 4c_4) H^2 \right]. \quad (73)$$

The coefficient of the mass term in the same de Sitter background is given by

$$\mathcal{M} = \frac{m^2 k^4}{\mathcal{D}_E^2} \left(\mathcal{M}_8 \tilde{k}^8 + \mathcal{M}_6 \tilde{k}^6 + \mathcal{M}_4 \tilde{k}^4 + \mathcal{M}_2 \tilde{k}^2 + \mathcal{M}_0 \right), \quad (74)$$

with

$$\mathcal{M}_8 = 4c_2^2 (c_3 + c_4), \quad (75)$$

$$\mathcal{M}_6 = c_2 (3c_1 - 12c_3 - 4c_4) (2c_1 - 5c_2 - 12c_3 - 20c_4) H^2 + c_1 c_2^2 (c_1 + 24c_3 + 40c_4) m^2, \quad (76)$$

$$\mathcal{M}_4 = 3 (3c_2 + 8c_4) (3c_1 - 12c_3 - 4c_4)^2 H^4 - 3c_2 (3c_1 - 12c_3 - 4c_4) [c_1 (3c_1 + 2c_2 + 32c_4) + 4c_2 (3c_3 + 5c_4)] H^2 m^2 + 6c_1^2 c_2^2 (c_1 + 6c_3 + 26c_4) m^4, \quad (77)$$

$$\mathcal{M}_2 = 9c_2 [(3c_1 - 12c_3 - 4c_4) H^2 - c_1^2 m^2] [(c_2 + 16c_4) (3c_1 - 12c_3 - 4c_4) H^2 - c_1 c_2 (c_1 + 32c_4) m^2] m^2, \quad (78)$$

$$\mathcal{M}_0 = 216 c_2^2 c_4 [(3c_1 - 12c_3 - 4c_4) H^2 - c_1^2 m^2]^2 m^4. \quad (79)$$

As in the vector sector, we require the kinetic coefficient \mathcal{K} to be strictly positive, for any value of k . Taking into account $c_2 > 0$, inspection of the expressions (72) and (73) shows that this is the case provided the two following conditions are satisfied (except in the case $c_1 = c_2$ discussed just below):

$$0 < c_1 < 2c_2, \quad c_1^2 m^2 + (-3c_1 + 12c_3 + 4c_4) H^2 > 0 \quad (c_1 \neq c_2). \quad (80)$$

In the special case $c_1 = c_2$, the kinetic coefficient \mathcal{K} is always positive (assuming $c_2 > 0$) since it reduces to

$$\mathcal{K} = \frac{m^2 k^4 c_2}{\tilde{k}^2 + 3m^2 c_2} \quad (c_1 = c_2). \quad (81)$$

The sound speed of the scalar mode can be read from the action (71), which gives

$$c_S^2 = -\frac{\mathcal{M}}{\mathcal{K}} \frac{a^2}{k^2}. \quad (82)$$

The full explicit expression of c_S^2 is lengthy and not very illuminating, but in the high momentum limit, $k^2/a^2 \gg m^2, H^2$, it reduces to

$$c_S^2 \simeq -\frac{4c_2(c_3 + c_4)}{c_1(2c_2 - c_1)} \quad (k^2/a^2 \gg m^2, H^2) \quad (83)$$

Therefore, taking into account the first condition in (80), we find that there is no gradient instability in the scalar sector, i.e. $c_S^2 > 0$, provided

$$c_3 + c_4 < 0. \quad (84)$$

In the limit $\mathcal{H} \equiv H/M \rightarrow 0$ in the theory with dilaton-like symmetry ($b(\Phi) = e^{M\Phi}$), we have

$$\begin{aligned} \frac{\mathcal{K}}{m^2 k^4} &\simeq \mathcal{H} \tilde{k}^{-2} \frac{\tilde{c}_1(-\tilde{c}_1 + 2\tilde{c}_2) \tilde{k}^2 + 3\tilde{c}_2(-3\tilde{c}_1 + 12\tilde{c}_3 + 4\tilde{c}_4) H^2}{\tilde{c}_2 \tilde{k}^2 + 3(-3\tilde{c}_1 + 12\tilde{c}_3 + 4\tilde{c}_4) H^2} \\ \frac{\mathcal{M}}{m^2 k^4} &\simeq \mathcal{H} \frac{4\tilde{c}_2^2(\tilde{c}_3 + \tilde{c}_4) \tilde{k}^4 + \tilde{c}_2(3\tilde{c}_1 - 12\tilde{c}_3 - 4\tilde{c}_4)(2\tilde{c}_1 - 5\tilde{c}_2 - 12\tilde{c}_3 - 20\tilde{c}_4) H^2 \tilde{k}^2 + 3(3\tilde{c}_2 + 8\tilde{c}_4)(3\tilde{c}_1 - 12\tilde{c}_3 - 4\tilde{c}_4)^2 H^4}{\left[\tilde{c}_2 \tilde{k}^2 + 3(-3\tilde{c}_1 + 12\tilde{c}_3 + 4\tilde{c}_4) H^2\right]^2} \end{aligned} \quad (85)$$

where $\tilde{c}_i \equiv c_i/\mathcal{H}$, assuming $\tilde{c}_i \sim \mathcal{O}(1)$ as discussed at the end of Subsection IV B. Thus in the limit $\mathcal{H} \rightarrow 0$, the scalar sector does not exhibit any pathological behaviors such as infinite or vanishing propagation speed, which would typically be an indicator of strong coupling at nonlinear level.

D. Summary of stability condition

In this subsection, we summarize the conditions for the perturbations to be stable and we reexpress them in term of the derivatives of the potential, via the relations (57). We recall that these conditions were derived by assuming a de Sitter solution in pure massive gravity, i.e. without matter. This implies in particular $\rho_g + P_g = 0$, giving the relation

$$2\mathcal{U}' + \frac{1}{N} (2\mathcal{E}' - \bar{\mathcal{E}}) = 0, \quad (86)$$

which we are going to use below to express $\bar{\mathcal{E}}$ in terms of \mathcal{U}' , \mathcal{E}' and \bar{N} .

As shown in the previous subsections, the tensor sector always has a healthy kinetic term and no gradient instability. Stability in the vector sector requires $c_2 > 0$, i.e.

$$2\mathcal{U}' + \frac{\mathcal{E}''}{N} > 0, \quad (87)$$

and, according to (61),

$$m_T^2 = 4m^2 \left(\mathcal{U}' + \frac{\mathcal{E}'}{N} + \mathcal{U}_t'' + \frac{\mathcal{E}_t''}{N} \right) > 0. \quad (88)$$

In the scalar sector, we need the two additional conditions (80) to avoid ghost instabilities, except if $c_1 = c_2$ i.e. $\mathcal{E}'' = 0$. These conditions translate into

$$2\mathcal{U}' - \frac{\mathcal{E}''}{\bar{N}} > 0, \quad \frac{m^2}{H^2}\mathcal{U}'^2 - 2\mathcal{U}' + \frac{\mathcal{E}'}{\bar{N}} - 6\left(\mathcal{U}_s'' + \frac{\mathcal{E}_s''}{\bar{N}}\right) - 2\left(\mathcal{U}_t'' + \frac{\mathcal{E}_t''}{\bar{N}}\right) > 0 \quad (\mathcal{E}'' \neq 0). \quad (89)$$

Finally, the high momentum sound speed (83) is given by

$$c_s^2 \simeq 2 \frac{\mathcal{U}' + \mathcal{E}'/\bar{N} + 2(\mathcal{U}_s'' + \mathcal{E}_s''/\bar{N} + \mathcal{U}_t'' + \mathcal{E}_t''/\bar{N})}{2\mathcal{U}' - \mathcal{E}''/\bar{N}} \quad (k^2/a^2 \gg m^2, H^2) \quad (90)$$

and therefore, the absence of gradient instability is guaranteed by the extra condition

$$\mathcal{U}' + \frac{\mathcal{E}'}{\bar{N}} + 2\left(\mathcal{U}_s'' + \frac{\mathcal{E}_s''}{\bar{N}} + \mathcal{U}_t'' + \frac{\mathcal{E}_t''}{\bar{N}}\right) > 0. \quad (91)$$

Putting everything together, for $\mathcal{E}'' \neq 0$, the following conditions must be satisfied simultaneously:

$$2\mathcal{U}' > \left| \frac{\mathcal{E}''}{\bar{N}} \right|, \quad (92)$$

$$\mathcal{U}' + \frac{\mathcal{E}'}{\bar{N}} + \mathcal{U}_t'' + \frac{\mathcal{E}_t''}{\bar{N}} > 0, \quad (93)$$

$$\mathcal{U}' + \frac{\mathcal{E}'}{\bar{N}} + 2\left(\mathcal{U}_s'' + \frac{\mathcal{E}_s''}{\bar{N}} + \mathcal{U}_t'' + \frac{\mathcal{E}_t''}{\bar{N}}\right) > 0, \quad (94)$$

$$\frac{m^2}{H^2}\mathcal{U}'^2 - 2\mathcal{U}' + \frac{\mathcal{E}'}{\bar{N}} - 6\left(\mathcal{U}_s'' + \frac{\mathcal{E}_s''}{\bar{N}}\right) - 2\left(\mathcal{U}_t'' + \frac{\mathcal{E}_t''}{\bar{N}}\right) > 0. \quad (95)$$

Let us mention the special case $\mathcal{E}'' = 0$, which corresponds to $c_1 = c_2$. In this case, the first condition above reduces to $\mathcal{U}' > 0$ and the last condition is no longer required.

Since generically the number of arbitrary functions exceeds the number of the constraint inequalities, it is naturally expected that there is a broad region in parameter space where the system is free from instabilities. In the next section, we will give an explicit example, which shows the existence of such stable solutions. When matter is present in the universe, the instability conditions are more involved, but matter will eventually be diluted and become negligible, so that the conditions derived above remain useful for the long-term stability.

V. EXPLICIT EXAMPLE

In this section we consider a simple explicit example, especially focusing on the behavior of the effective gravitational field which is induced by the graviton mass term. We shall see that there actually exists a healthy cosmological solution satisfying all conditions for healthy perturbations derived in the previous section.

A. Ansatz for the potential

In the simple example that we investigate in this section, the explicit forms of \mathcal{U} and \mathcal{E} are assumed to be

$$\mathcal{U}(\mathcal{K}^{IJ}, f_{IJ}) = u_0 + u_1 \mathcal{K}^{IJ} f_{IJ} + \frac{1}{2} \left[u_{2s} (\mathcal{K}^{IJ} f_{IJ})^2 + u_{2q} \mathcal{K}^{IJ} f_{JK} \mathcal{K}^{KL} f_{LI} \right], \quad (96)$$

$$\mathcal{E}(\Gamma^{IJ}, \xi^I, f_{IJ}) = v_0 + v_1 \Gamma^{IJ} f_{IJ} + \frac{1}{2} \left[v_{2s} (\Gamma^{IJ} f_{IJ})^2 + v_{2q} \Gamma^{IJ} f_{JK} \Gamma^{KL} f_{LI} \right] + \frac{1}{2} w_2 \xi^I \xi^J f_{IJ}, \quad (97)$$

where the coefficients u_0, u_1 , etc. are constants.

In terms of the ratio $X \equiv b/a$, introduced in (33), we can write the background values of \mathcal{U} and \mathcal{E} as

$$\bar{\mathcal{U}} = u_0 + 3u_1 X^2 + \frac{3}{2} u_2 X^4, \quad \bar{\mathcal{E}} = v_0 + 3v_1 X^2 + \frac{3}{2} v_2 X^4, \quad (98)$$

with

$$u_2 \equiv 3u_{2s} + u_{2q}, \quad v_2 \equiv 3v_{2s} + v_{2q}. \quad (99)$$

Following the definitions (35) and (50-51) of the various parameters derived from the potential, we obtain explicitly

$$\begin{aligned}\mathcal{U}' &= u_1 X^2 + u_2 X^4, & \mathcal{U}_s'' &= u_{2s} X^4, & \mathcal{U}_t'' &= u_{2q} X^4, \\ \mathcal{E}' &= v_1 X^2 + v_2 X^4, & \mathcal{E}_s'' &= v_{2s} X^4, & \mathcal{E}_t'' &= v_{2q} X^4, & \mathcal{E}'' &= w_2 X^2.\end{aligned}\quad (100)$$

B. Background equations

Following (37), the background equations are given by

$$3M_{\text{Pl}}^2 H^2 = \rho_g + \rho_m, \quad M_{\text{Pl}}^2 \left(2 \frac{\dot{H}}{H} + 3H^2 \right) = -(P_g + P_m), \quad (101)$$

where ρ_m and P_m represent the energy density and pressure from usual matter and ρ_g and P_g are those from the effective gravitational field. In the current model, their explicit forms are given as

$$\rho_g = M_{\text{Pl}}^2 m^2 \left(u_0 + 3u_1 X^2 + \frac{3}{2} u_2 X^4 \right), \quad (102)$$

and

$$P_g = M_{\text{Pl}}^2 m^2 \left[-u_0 - u_1 X^2 + \frac{1}{2} u_2 X^4 + \frac{1}{N} \left(-v_0 - v_1 X^2 + \frac{1}{2} v_2 X^4 \right) \right]. \quad (103)$$

C. De Sitter solution

In the absence of matter ($\rho_m = P_m = 0$) and assuming the dilaton-like symmetry ($\dot{b}/b = M$), X and H are determined by the algebraic equations (44) and (45). One can instead solve these equations with respect to two parameters of the model, e.g. u_0 and v_0 , as

$$u_0 = \frac{3H^2}{m^2} - 3u_1 X^2 - \frac{3}{2} u_2 X^4, \quad v_0 = 2 \frac{M}{H} (u_1 X^2 + u_2 X^4) - v_1 X^2 + \frac{1}{2} v_2 X^4. \quad (104)$$

One can also solve (100) with respect to $(u_1, u_{2s}, u_{2q}, v_1, v_{2s}, v_{2q}, w_2)$ to express them in terms of $(\mathcal{U}', \mathcal{U}_s'', \mathcal{U}_t'', \mathcal{E}', \mathcal{E}_s'', \mathcal{E}_t'', \mathcal{E}'')$ and X . Hereafter we thus consider $(H, X, \mathcal{U}', \mathcal{U}_s'', \mathcal{U}_t'', \mathcal{E}', \mathcal{E}_s'', \mathcal{E}_t'', \mathcal{E}'')$ as independent parameters. Since X is constant, \bar{N} is determined from (41) (with $\dot{\mathcal{U}} = 0$ and $\dot{b}/b = M$) as

$$\bar{N} = \frac{M}{H}. \quad (105)$$

With $\dot{b}/b = M$, the stability condition summarized in subsection IV D become

$$2\mathcal{U}' > |\mathcal{E}''| \frac{H}{M}, \quad (106)$$

$$\mathcal{U}' + \mathcal{U}_t'' + (\mathcal{E}' + \mathcal{E}_t'') \frac{H}{M} > 0, \quad (107)$$

$$\mathcal{U}' + 2\mathcal{U}_s'' + 2\mathcal{U}_t'' + (\mathcal{E}' + 2\mathcal{E}_s'' + 2\mathcal{E}_t'') \frac{H}{M} > 0, \quad (108)$$

$$\frac{m^2}{H^2} \mathcal{U}'^2 - 2(\mathcal{U}' + 3\mathcal{U}_s'' + \mathcal{U}_t'') + (\mathcal{E}' - 6\mathcal{E}_s'' - 2\mathcal{E}_t'') \frac{H}{M} > 0, \quad (109)$$

assuming $M > 0$. As there are 9 independent parameters in the 4 constraint inequalities, none of which is conflicting with each other, this suffices to show the stability of the present example. This confirms the existence of stable, pure de Sitter solutions in this theory.

VI. COMPARISON WITH PREVIOUS RESULTS

A. dRGT

As briefly reviewed in the introduction, the dRGT theory with Minkowski fiducial metric allows for self-accelerating open FLRW solutions [14]. If we replace the fiducial metric with a de Sitter or FLRW one, then the theory permits self-accelerating flat/closed/open FLRW solutions. However, for those solutions the quadratic action for perturbations does not contain kinetic terms for three among five degrees of freedom [15]. This is the origin of the nonlinear ghost instability found in [19]. Based on this nonlinear instability and the linear instability called Higuchi ghost [20], it was argued that all FLRW solutions in the dRGT theory are unstable.

The analysis in Section IV can reproduce the key result of [15], namely the vanishing of kinetic terms for three among the five degrees of freedom for the perturbations. The quadratic action for the perturbations around self-accelerating solutions in the dRGT theory has a peculiar structure. As shown in an appendix of [15], it is in the general form (56) but with

$$c_0 = c_1 = c_2 = 0, \quad c_3 + c_4 = 0, \quad (\text{dRGT self-accelerating branch}). \quad (110)$$

Among the properties (110), $c_0 = 0$ always holds in theories without BD ghost. On the other hand, $c_1 = c_2 = 0$ is the origin of the vanishing kinetic terms. Indeed, $c_2 = 0$ already implies that μ^2 defined in (64) vanishes and thus the kinetic term for E_i in the vector action (66) vanishes. The coefficient \mathcal{K} defined in (72) also vanishes when $c_1 = c_2 = 0$, meaning that the kinetic term of E in the scalar action (71) vanishes.

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In [44], the fiducial metric was supposed to be strictly euclidean, i.e. without the scale factor b that we have introduced in (20). Their results can thus be reproduced by simply setting $\dot{b}/b = 0$ in our equations. Indeed, in this case, (42) implies the constraint

$$H(2\mathcal{E}' - \bar{\mathcal{E}}) = 0. \quad (111)$$

For a generic choice of the function \mathcal{E} , if $H \neq 0$ then this equation would fix $X = b/a$ to a constant, which is a root of $2\mathcal{E}' - \bar{\mathcal{E}}$. However, since b is constant here, $X = \text{const.}$ would mean $H = 0$. To get a non-trivial cosmology, the function \mathcal{E} is thus severely restricted (so that $2\mathcal{E}' - \bar{\mathcal{E}}$ vanishes identically) and the effective gravitational energy density and pressure reduce to

$$\rho_g \equiv M_{\text{Pl}}^2 m^2 \bar{\mathcal{U}}, \quad P_g \equiv M_{\text{Pl}}^2 m^2 (2\mathcal{U}' - \bar{\mathcal{U}}). \quad (112)$$

In this situation, an equation of state $P_g = -\rho_g$, corresponding to a de Sitter solution, necessarily requires

$$\mathcal{U}' = 0 \quad (\text{de Sitter}). \quad (113)$$

This implies that the coefficients c_1 and c_2 defined in (57) vanish. As explained in the previous subsections, this inevitably means that the vector and scalar modes have vanishing kinetic terms, in agreement with the results of [44]. In that work, in order to avoid this pathological behaviour, they introduce a slight deviation from the equation of state $P_g = -\rho_g$. By contrast, in our case, the extra freedom due to the scale factor b enables us to find strictly de Sitter solutions that are not pathological.

VII. SUMMARY AND DISCUSSIONS

In this work, we have investigated FLRW cosmological solutions and their stability in a class of massive gravity theories with five degrees of freedom. The theories we have considered for this study are those that respect the 3-dimensional maximal symmetry in the space of Stückelberg fields, but not necessarily the 4-dimensional one, in accord with the symmetry of FLRW cosmology [42, 43]. In any theory of massive gravity written in the unitary gauge, diffeomorphism invariance is broken by the graviton mass term at cosmological scales. It is thus rather natural to suppose that the low energy effective field theory of massive gravity respects only the symmetry of the cosmological background, i.e. the 3-dimensional maximal symmetry.

After a detailed study of linear perturbations, we have shown that, unlike in the dRGT theory, all five degrees of freedom around self-accelerating FLRW backgrounds generically have non-vanishing kinetic terms in this theory. This

means that the setup developed in the present paper evades the no-go result for massive gravity cosmology found recently in [19].

It is important to stress that our model differs not only from the dRGT theory but also from that studied in a previous work on the cosmology of rotation-invariant massive gravity [44]. In that previous study, it was found that the kinetic terms of three among five degrees of freedom vanish on de Sitter backgrounds. By contrast, in the model considered in the present paper, all five degrees of freedom generically have non-vanishing kinetic terms on de Sitter as well as non-de Sitter FLRW backgrounds. The main difference between our theoretical setup and the previous one in [44] is that we allow the graviton mass term to depend on the temporal Stückelberg field in a non-trivial way. In the dRGT limit, such a non-trivial dependence manifests itself as a de Sitter or FLRW fiducial metric. (Note however that three among the five degrees of freedom have vanishing kinetic terms in any self-accelerating FLRW backgrounds in dRGT with any FLRW fiducial metric.) For illustration, we have also presented simple models that allow for stable de Sitter solutions. Although we have restricted our analysis, for simplicity, to cases assuming an additional dilaton-like symmetry in the Stückelberg field space, our results correspond to a generic feature of rotation-invariant massive gravity theory with a non-trivial fiducial metric.

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Appendix A: Quadratic expansion of Einstein-Hilbert action

In this appendix, we give some details about the expansion of the Einstein-Hilbert action up to quadratic order in linear perturbations around a spatially flat FLRW background solution.

We employ the ADM formalism in which the metric is given by (22). The Einstein-Hilbert action can then be written in the form

$$S_{\text{EH}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x R = \frac{M_{\text{Pl}}^2}{2} \int d^4x N \sqrt{\gamma} \left(K_{ij} K^{ij} - K^2 + {}^{(3)}R \right), \quad (\text{A1})$$

where $\gamma = \det(\gamma_{ij})$, ${}^{(3)}R$ is the Ricci scalar associated with the 3-D spatial metric γ_{ij} , and

$$K_{ij} = \frac{1}{2N} (\dot{\gamma}_{ij} - N_{i|j} - N_{j|i}) \quad (\text{A2})$$

is the extrinsic curvature tensor (the symbol $|$ denotes the spatial covariant derivative associated with the spatial metric γ_{ij}). Here the indices on K_{ij} are raised (lowered) by γ^{ij} (γ_{ij}), and $K \equiv K^i_i$.

We now expand the metric around the flat FLRW background. The metric perturbations are described by the lapse perturbation δN , the shift N^i and the perturbations h_{ij} of the three-dimensional metric, as described at the beginning of Section IV. To compute the Einstein-Hilbert action up to quadratic order, one needs to compute K_j^i up to linear order and ${}^{(3)}R$ up to quadratic order, as is clear from (A1). One can write, up to quadratic order,

$$K_{ij} K^{ij} - K^2 = 6H^2 - 4HK + \delta K_j^i \delta K_i^j - \delta K^2. \quad (\text{A3})$$

Moreover, by integration by parts, one has

$$\int d^4x \sqrt{-g} HK = \int d^4x \sqrt{-g} H \nabla_\mu n^\mu \rightarrow - \int d^4x \sqrt{-g} \frac{\dot{H}}{N}, \quad (\text{A4})$$

where n^μ is the unit vector normal to the constant time hypersurfaces. Therefore, the Einstein-Hilbert action can be rewritten, up to total derivatives, as

$$S_{\text{EH}} = M_{\text{Pl}}^2 \int d^4x \sqrt{-g} (3H^2) + \tilde{S}_{\text{EH}}$$

$$\tilde{S}_{\text{EH}} \equiv 2M_{\text{Pl}}^2 \int d^4x \sqrt{\gamma} \dot{H} + \frac{M_{\text{Pl}}^2}{2} \int d^4x N \sqrt{\gamma} \left(\delta K_j^i \delta K_i^j - \delta K^2 + {}^{(3)}R \right), \quad (\text{A5})$$

where the explicit expressions are given by

$$\begin{aligned} \delta K_j^i &= \frac{1}{2N} \left(\dot{h}_j^i - 2H\delta N \delta_j^i - \delta^{ik} \delta_{jl} \partial_k N^l - \partial_j N^i \right) \\ \delta K &= \frac{1}{2N} \left(\dot{h} - 3H\delta N - 2\partial_k N^k \right) \\ {}^{(3)}R &= \frac{1}{a^2} \left(-\partial^2 h + \partial_i \partial_j h^{ij} + h^{ij} \partial^2 h_{ij} + h^{ij} \partial_i \partial_j h - 2h^{ij} \partial_i \partial^k h_{kj} + \frac{3}{4} \partial_i h_{jk} \partial^i h^{jk} \right. \\ &\quad \left. - \frac{1}{2} \partial_k h_{ij} \partial^j h^{ik} - \partial_i h^{ij} \partial^k h_{kj} - \frac{1}{4} \partial^k h \partial_k h + \partial_i h^{ij} \partial_j h \right), \end{aligned} \quad (\text{A6})$$

In the above expressions and in the following, the spatial indices are raised and lowered by δ^{ij} and δ_{ij} , respectively. Substituting the above expressions into (A5) and using the following formula valid up to the quadratic order,

$$\sqrt{\gamma} = a^3 \left(1 + \frac{1}{2}h - \frac{1}{4}h_{ij}h^{ij} + \frac{1}{8}h^2 \right), \quad (\text{A7})$$

one finally obtains the quadratic action for the Einstein-Hilbert term in the form

$$\begin{aligned} \tilde{S}_{\text{EH}}^{(2)} &= M_{\text{Pl}}^2 \int d^4x \bar{N} a^3 \left\{ \frac{1}{8\bar{N}^2} \dot{h}_{ij} \dot{h}^{ij} - \frac{1}{8\bar{N}^2} \dot{h}^2 + \frac{H}{\bar{N}^2} \delta N \dot{h} - \frac{1}{2\bar{N}} \dot{h}_{ij} \partial^i N^j + \frac{1}{2\bar{N}} \dot{h} \partial_k N^k - 3H^2 \frac{\delta N^2}{\bar{N}^2} \right. \\ &\quad + \frac{1}{4\bar{N}^2} \partial_i N_j (\partial^i N^j + \partial^j N^i) - \frac{2H}{\bar{N}^2} \delta N \partial_i N^i - \frac{1}{2\bar{N}^2} (\partial_k N^k)^2 + \frac{1}{a^2} \left[\left(\frac{\delta N}{\bar{N}} + \frac{1}{2}h \right) (-\partial^2 h + \partial_i \partial_j h^{ij}) \right. \\ &\quad + h^{ij} \partial^2 h_{ij} + h^{ij} \partial_i \partial_j h - 2h^{ij} \partial_i \partial^k h_{kj} + \frac{3}{4} \partial_i h_{jk} \partial^i h^{jk} - \frac{1}{2} \partial_k h_{ij} \partial^j h^{ik} - \partial_i h^{ij} \partial^k h_{kj} - \frac{1}{4} \partial^k h \partial_k h + \partial_i h^{ij} \partial_j h \left. \right] \\ &\quad \left. + 2\frac{\dot{H}}{\bar{N}} \left(\frac{1}{8}h^2 - \frac{1}{4}h_{ij}h^{ij} \right) \right\}. \end{aligned} \quad (\text{A8})$$

We now decompose this action into its tensor, vector and scalar components.

a. Tensor sector

The lapse and shift are unperturbed and h_{ij} is transverse and traceless:

$$\partial^i h_{ij} = 0, \quad h = 0. \quad (\text{A9})$$

The quadratic part of the Einstein-Hilbert action for tensor modes is

$$\tilde{S}_{\text{EH}}^{(2)T} = \int d^4x \bar{N} a^3 \left(\frac{1}{8\bar{N}^2} \dot{h}_{ij} \dot{h}^{ij} - \frac{1}{8a^2} \partial_k h_{ij} \partial^k h^{ij} - \frac{\dot{H}}{2} h_{ij} h^{ij} \right). \quad (\text{A10})$$

b. Vector sector

The only perturbations are N^i and

$$h_{ij} = \partial_i E_j + \partial_j E_i, \quad (\text{A11})$$

with

$$\partial_i N^i = 0, \quad \partial_i E^i = 0. \quad (\text{A12})$$

Substituting these into the action (A8), and simplifying via integrations by parts, one finally obtains

$$\tilde{S}_{\text{EH}}^{(2)V} = \int d^4x \frac{a^3}{N} \left(\frac{1}{4} \partial_i \dot{E}_j \partial^j \dot{E}^i + \frac{1}{4} \partial_i N_j \partial^i N^j - \frac{1}{2} \partial_i \dot{E}_j \partial^i N^j - \bar{N} \dot{H} \partial_i E_j \partial^i E^j \right). \quad (\text{A13})$$

c. Scalar sector

The metric perturbations are described by δN , and

$$N^i = \partial^i B, \quad h_{ij} = 2C\delta_{ij} + 2\partial_i \partial_j E. \quad (\text{A14})$$

After a lot of simplifications, one finds

$$\begin{aligned} \tilde{S}_{\text{EH}}^{(2)S} = \int d^4x a^3 & \left\{ \frac{3}{2} \dot{C}^2 + \dot{C} \partial^2 \dot{E} + \frac{1}{2} (\partial^2 \dot{E})^2 + \frac{3}{2} H^2 \delta N^2 - 3H \delta N \dot{C} - H \delta N \partial^2 \dot{E} + H \delta N \partial^2 B + \frac{1}{2} (\partial^2 B)^2 \right. \\ & \left. - \dot{C} \partial^2 B - \partial^2 \dot{E} \partial^2 B + \frac{1}{2a^2} \left[-4\partial^2 C \delta N + 2(\partial C)^2 \right] + \dot{H} \left[3C^2 + 2C \partial^2 E - (\partial^2 E)^2 \right] \right\}. \end{aligned} \quad (\text{A15})$$

Appendix B: Condition for no BD ghost at quadratic order

In this appendix, we extend our analysis of the linear perturbations to include the case of an arbitrary coefficient c_0 in (56). We then show that a non-zero c_0 leads to the presence of a sixth propagating degree of freedom, which is known to lead to a BD ghost at the nonlinear level. The fact that the potential (11) leads to $c_0 = 0$ is thus consistent with the property that this potential is free of the BD-ghost at all orders [42, 43]. Here we restrict our consideration to the flat FLRW background without matter, as the inclusion of matter leads to complicated expressions, making our analysis less transparent.⁷

We thus start from (58) with (A8) and (56), without any restriction on the coefficients c_i . They can also depend on time, in contrast with our assumption in Section IV. We then decompose the perturbations into tensor, vector and scalar modes as in (59), (62) and (70), respectively. The actions for the tensor and vector modes are identical to those derived in Subsections IV A and IV B and the stability conditions in these sectors are thus unchanged.

By contrast, the action in the scalar sector is different in general. After integrating out the non-dynamical lapse δN and shift component B , the scalar quadratic action contains two degrees of freedom and can be written in the following matrix form

$$S_S^{(2)} = \int dt d^3k a^3 \left[\dot{\delta}^\dagger \tilde{T} \dot{\delta} + \left(\delta^\dagger \tilde{X} \delta + \text{h.c.} \right) + \delta^\dagger \tilde{\Omega}^2 \delta \right] \quad (\text{B1})$$

where $\delta \equiv (C, E)^T$, the first two coefficient matrices are given by

$$\tilde{T} = \frac{m^2}{\left(\frac{k^2}{a^2} + 3m^2 c_2 \right) H^2 - m^4 c_0 c_2} \begin{bmatrix} \left(\frac{k^2}{a^2} + 3m^2 c_2 \right) c_0 & -m^2 k^2 c_0 c_2 \\ -m^2 k^2 c_0 c_2 & k^4 H^2 c_2 \end{bmatrix} \quad (\text{B2})$$

$$\tilde{X} = \begin{bmatrix} 0 & m^2 k^2 H \frac{\frac{k^2}{a^2} (c_1 - c_2)}{\left(\frac{k^2}{a^2} + 3m^2 c_2 \right) H^2 - m^4 c_0 c_2} \\ 0 & 0 \end{bmatrix} \quad (\text{B3})$$

and $\tilde{\Omega}^2$ is a 2×2 Hermitian matrix, whose rather lengthy expression is not needed for our current purpose.

In general, the above action contains two dynamical degrees of freedom. It is known that one of these scalar modes leads to ghost-like instabilities at the nonlinear level. To get rid of this pathological mode, one must ensure that only

⁷ We have verified that the condition for the absence of the BD ghost ($c_0 = 0$) is unchanged by including matter.

one dynamical degree of freedom appears in the scalar action. A necessary condition is that the determinant of the matrix \tilde{T} , given in (B2), vanishes, namely

$$\det \tilde{T} = m^4 k^4 \frac{c_0 c_2}{\left(\frac{k^2}{a^2} + 3m^2 c_2\right) H^2 - m^4 c_0 c_2} = 0. \quad (\text{B4})$$

Since stability in the vector sector requires $c_2 > 0$, this imposes the condition

$$c_0 = 0. \quad (\text{B5})$$

Once this condition is imposed, one can check that the scalar mode C becomes non-dynamical. After integrating it out, one recovers the scalar quadratic action (71) with only one dynamical degree of freedom, E .

In view of the above discussion, it appears natural that the theory considered in the main text leads to $c_0 = 0$, as explicitly shown in (57). This is a non-trivial consistency check for the theory, which should be free of the BD ghost at all orders [42, 43].

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